

A NEW PROOF FOR THE EXISTENCE OF AN EQUIVARIANT ENTIRE SOLUTION CONNECTING THE MINIMA OF THE POTENTIAL FOR THE SYSTEM $\Delta u - W_u(u) = 0$

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ABSTRACT. Recently, Giorgio Fusco and the author in [1] studied the system $\Delta u - W_u(u) = 0$ for a class of potentials that possess several global minima and are invariant under a general finite reflection group, and established existence of equivariant solutions connecting the minima in certain directions at infinity, together with an estimate. In this paper a new proof is given which, in particular, avoids the introduction of a pointwise constraint in the minimization process.

1. INTRODUCTION

The study of the system

$$(1) \quad \Delta u - W_u(u) = 0, \text{ for } u : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

where $W : \mathbb{R}^n \rightarrow \mathbb{R}$ and $W_u := (\partial W / \partial u_1, \dots, \partial W / \partial u_n)^\top$, under symmetry hypotheses on the potential W was initiated in Bronsard, Gui, and Schatzman [3], where existence for the case $n = 2$ with the symmetries of the equilateral triangle was settled. About twelve years later this work was followed by Gui and Schatzman [11], where the case $n = 3$ for the symmetry group of tetrahedron was established. The corresponding solutions are known as the *triple junction* and the *quadruple junction* respectively. This class of solutions is characterized by the fact that they *connect* the N global minima of the potential W , that is,

$$(2) \quad \lim_{\lambda \rightarrow +\infty} u(\lambda \eta_i) = a_i, \text{ for } i = 1, \dots, N,$$

for certain unit vectors $\eta_i \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ is the unit sphere. These solutions are related to minimal surface complexes, and particularly to the singular points there (see Taylor [17], Dierkes *et al.* [5, 6]) via the blow-down limit $u_\varepsilon(x) := u(x/\varepsilon)$ (see Baldo [2]). Recently in [1] certain general hypotheses on W were identified and the problem was settled for general dimension n and for any reflection group G on \mathbb{R}^n .

In this paper we want to give a new derivation of this result, which is based on a positivity property of the gradient flow associated to (1) and comparison arguments involving subharmonic functions, ingredients already existing in [1], but now supplemented with a Kato-type inequality and the De Giorgi oscillation lemma. The present paper is self-contained. Our hope is that this simpler proof will be more adaptable to the general case of a potential W without symmetry requirements. In order to bring out clearly the underlying ideas, we refrain from any generalization which could complicate the technical part.

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1.1. Notation. We denote by B_R the ball of radius $R > 0$ centered at the origin and by $W_E^{1,2}(B_R; \mathbb{R}^n)$ the subspace of *equivariant maps*, that is, $u(gx) = gu(x)$, for all $g \in G$ and $x \in \mathbb{R}^n$. We also denote by $\langle \cdot, \cdot \rangle$ the Euclidean inner product, by $|\cdot|$ the Euclidean norm, and by $d(x, \partial D)$ the distance of x from ∂D . In the case of finite groups G , the notation $|G|$ stand for the number of elements of the group.

We denote the functional associated to (1) by

$$(3) \quad J(u) = \int_{\mathbb{R}^n} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx.$$

A *Coxeter group* is a finite subgroup of the orthogonal group $O(\mathbb{R}^n)$, generated by a set of reflections. A *reflection* $\gamma \in G$ is associated to the hyperplane

$$\pi_\gamma = \{x \in \mathbb{R}^n \mid \langle x, \eta_\gamma \rangle = 0\},$$

via

$$\gamma x = x - 2\langle x, \eta_\gamma \rangle \eta_\gamma, \text{ for } x \in \mathbb{R}^n,$$

where $\eta_\gamma \in \mathbb{S}^{n-1}$ is a unit vector. Every finite subgroup of $O(\mathbb{R}^n)$ has a *fundamental region*¹, that is, a subset $F \subset \mathbb{R}^n$ with the following properties:

- (i) F is open and convex,
- (ii) $F \cap gF = \emptyset$, for $I \neq g \in G$, where I is the identity,
- (iii) $\mathbb{R}^n = \cup \{g\overline{F} \mid g \in G\}$.

We choose the orientation of η_γ so that $F \subset \mathcal{P}_\gamma^+$, where $\mathcal{P}_\gamma^+ = \{x \in \mathbb{R}^n \mid \langle x, \eta_\gamma \rangle > 0\}$. Then, we have

$$(4) \quad F = \cap_{\gamma \in \Gamma} \mathcal{P}_\gamma^+,$$

where $\Gamma \subset G$ is the set of all reflections in G . Given $a \in G$, the *stabilizer* of a , denoted by G_a , is the subgroup of G that fixes a .

1.2. The theorem ([1]). We begin with the hypotheses.

Hypothesis 1 (N nondegenerate global minima). *The potential W is of class C^2 and satisfies $W(a_i) = 0$, for $i = 1, \dots, N$, and $W > 0$ on $\mathbb{R}^n \setminus \{a_1, \dots, a_N\}$. Furthermore, there holds $v^\top \partial^2 W(u) v \geq 2c^2 |v|^2$, for $v \in \mathbb{R}^n$ and $|u - a_i| \leq \bar{q}$, for some $c, \bar{q} > 0$, and for $i = 1, \dots, N$.*

Hypothesis 2 (Symmetry). *The potential W is invariant under a finite reflection group G acting on \mathbb{R}^n (Coxeter group), that is,*

$$(5) \quad W(gu) = W(u), \text{ for all } g \in G \text{ and } u \in \mathbb{R}^n.$$

Moreover, we assume that there exists $M > 0$ such that $W(su) \geq W(u)$, for $s \geq 1$ and $|u| = M$.

We seek *equivariant* solutions of system (1), that is, solutions satisfying

$$(6) \quad u(gx) = gu(x), \text{ for all } g \in G \text{ and } x \in \mathbb{R}^n.$$

Hypothesis 3 (Location and number of global minima). *Let $F \subset \mathbb{R}^n$ be a fundamental region of G . We assume that \overline{F} (the closure of F) contains a single global minimum of W , say a_1 , and let G_{a_1} be the subgroup of G that leaves a_1 fixed. Then, as it follows by the invariance of W , the number of the minima of W is*

$$(7) \quad N = \frac{|G|}{|G_{a_1}|}.$$

¹See [10] or [14].

Hypothesis 4 (Q -monotonicity). *We restrict ourselves to potentials W for which there is a continuous function $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies*

$$(8) \quad Q(u + a_1) = |u| + H(u),$$

where $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is a C^2 function such that $H(0) = 0$ and $H_u(0) = 0$, and

$$(9a) \quad Q \text{ is convex,}$$

$$(9b) \quad Q(gu) = Q(u), \text{ for } u \in \mathbb{R}^n, \ g \in G_{a_1},$$

$$(9c) \quad Q(u + a_1) = |u| + H(u), \text{ in a neighborhood of } u = 0,$$

$$(9d) \quad Q(u) > 0, \text{ on } \mathbb{R}^n \setminus \{a_1\},$$

and, moreover,

$$(10) \quad \langle Q_u(u), W_u(u) \rangle \geq 0, \text{ in } D \setminus \{a_1\},$$

where we have set

$$(11) \quad D := \text{Int} \left(\bigcup_{g \in G_{a_1}} g\overline{F} \right).$$

Theorem 1.1 ([1]). *Under Hypotheses 1–4, there exists an equivariant classical solution to system (1) such that*

- (i) $|u(x) - a_1| \leq Ke^{-kd(x, \partial D)}$, for $x \in D$ and for positive constants k, K ,
- (ii) $u(\overline{F}) \subset \overline{F}$ and $u(D) \subset D$.

In particular, u connects the $N = |G|/|G_{a_1}|$ global minima of W in the sense that

$$\lim_{\lambda \rightarrow +\infty} u(\lambda g\eta) = ga_1, \text{ for all } g \in G,$$

uniformly for η in compact subsets of $D \cap \mathbb{S}^{n-1}$.

2. THE EXTENDED KATO INEQUALITY

We begin by presenting a straightforward extension of the classical Kato inequality. We follow the presentation in [13, p. 85]. Let $\hat{Q} : \mathbb{R}^m \rightarrow \mathbb{R}$ be a continuous function satisfying the following assumptions.

- (i) \hat{Q} is convex,
- (ii) $\hat{Q} > 0$ and $\hat{Q}_u \neq 0$, for $u \neq 0$,
- (iii) $\hat{Q} = |u| + H(u)$, for a C^2 function $H : \mathbb{R}^m \rightarrow \mathbb{R}$, such that $H(0) = 0$ and $H_u(0) = 0$.

Lemma 2.1. *Let $u \in L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ and suppose² that the distributional Laplacian $\Delta u \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$. Then,*

$$(12) \quad \Delta \hat{Q}(u) \geq \langle \Delta u, \hat{Q}_u(u) \rangle,$$

in the distributional sense, with the definition

$$\hat{Q}_u(u) := \begin{cases} \nabla_u \hat{Q}(u), & \text{for } u \neq 0, \\ 0, & \text{for } u = 0. \end{cases}$$

²The fact that u should be in $u \in L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ was pointed out to us by Panagiotis Smyrnelis. If H is assumed globally Lipschitz, then $u \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ suffices.

Remarks. The well-known Kato inequality for functions $u \in L^1_{\text{loc}}(\mathbb{R}^n; \mathbb{C})$ states that

$$(13) \quad \Delta|u| \geq \text{Re}[(\text{sgn } u)\Delta u],$$

in the distributional sense. The choice $|u| = \sqrt{u\bar{u}}$ and

$$\text{sgn } u = \begin{cases} 0, & \text{for } u = 0, \\ \bar{u}/|u|, & \text{for } u \neq 0, \end{cases}$$

is a special case, for $\hat{Q}(u) = |u|$ and $\text{Re}[u\bar{v}] = \langle u, v \rangle$.

Also, under the hypothesis $u \in W^{1,2}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m) \cap L^\infty(\mathbb{R}^n; \mathbb{R}^m)$ we note that $\hat{Q}(u(\cdot)) \in W^{1,2}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$ (see [15, p. 54] or [7, p. 130]). Therefore, (12) holds in $W^{1,2}_{\text{loc}}(\mathbb{R}^n; \mathbb{R}^m)$.

Proof. We utilize the summation convention. We first establish

$$(14) \quad \Delta\hat{Q}(u) \geq \langle \Delta u, \hat{Q}_u(u) \rangle,$$

for $u \in C^\infty$, except where $\hat{Q}(u)$ is not differentiable. Set

$$\hat{Q}_\varepsilon(u(x)) = \sqrt{\hat{Q}^2(u(x)) + \varepsilon^2}, \text{ for } \varepsilon > 0;$$

then,

$$(15) \quad \hat{Q}_\varepsilon \hat{Q}_{\varepsilon,i} = \hat{Q} \hat{Q}_{,u_k} u_{k,i}, \text{ where } \hat{Q}_{\varepsilon,i} := \frac{\partial}{\partial x_i} \hat{Q}_\varepsilon(u(x)),$$

and

$$(16) \quad (\hat{Q}_{\varepsilon,i})^2 = \left(\frac{\hat{Q}}{\hat{Q}_\varepsilon} \right)^2 (\hat{Q}_{,u_k} u_{k,i})^2 \leq (\hat{Q}_{,u_k} u_{k,i})^2.$$

Hence,

$$(17) \quad \hat{Q}_{\varepsilon,i} \hat{Q}_{\varepsilon,i} \leq (\hat{Q}_{,u_k} u_{k,i})(\hat{Q}_{,u_k} u_{k,i}),$$

therefore

$$(18) \quad |\nabla_x \hat{Q}_\varepsilon|^2 \leq |(\nabla_x u)^\top \hat{Q}_u|^2.$$

Moreover,

$$(19) \quad (\hat{Q}_\varepsilon \hat{Q}_{\varepsilon,i})_{,i} = \langle \Delta u, \hat{Q} \hat{Q}_u \rangle + \hat{Q} \langle (\partial^2 \hat{Q}) u_{,i}, u_{,i} \rangle + |(\nabla_x u)^\top \hat{Q}_u|^2,$$

where $\partial^2 \hat{Q}$ is the Hessian of \hat{Q} and $u_{,i} = (u_{1,i}, \dots, u_{m,i})$. By convexity it follows that

$$|\nabla_x \hat{Q}_\varepsilon|^2 + \hat{Q}_\varepsilon \Delta \hat{Q}_\varepsilon \geq \langle \Delta u, \hat{Q} \hat{Q}_u \rangle + |(\nabla_x u)^\top \hat{Q}_u|^2,$$

from which, by (18),

$$(20) \quad \Delta \hat{Q}_\varepsilon \geq \left\langle \Delta u, \frac{\hat{Q} \hat{Q}_u}{\hat{Q}_\varepsilon} \right\rangle.$$

At points of smoothness we can take the limit $\varepsilon \rightarrow 0$ and obtain (14).

We proceed by mollification. Let $w \in C^\infty(\mathbb{R}^n)$, with $w \geq 0$ and $\int w(x) dx = 1$. For $\delta > 0$ we define $w_\delta(x) = \delta^{-n} w(\delta^{-1}x)$ and set

$$I_\delta u := w_\delta * u, \text{ for } \delta > 0.$$

Then, $I_\delta u \rightarrow u$ and $\Delta(I_\delta u) \rightarrow \Delta u$ in L^1 , as $\delta \rightarrow 0$. Applying (20) to $I_\delta u$ we have

$$(21) \quad \Delta \hat{Q}_\varepsilon(I_\delta u) \geq \left\langle \Delta(I_\delta u), \frac{\frac{\partial}{\partial u}(\frac{1}{2}\hat{Q}^2)(I_\delta u)}{\hat{Q}_\varepsilon(I_\delta u)} \right\rangle.$$

Taking $\delta \rightarrow 0$ and utilizing that \hat{Q}^2 is everywhere differentiable and that the fraction inside the inner product in (21) is bounded (L^∞ requirement for $u(\cdot)$), by the dominated convergence theorem we have

$$(22) \quad \Delta \hat{Q}_\varepsilon(u) \geq \left\langle \Delta u, \frac{\frac{\partial}{\partial u}(\frac{1}{2}\hat{Q}^2)(u)}{\hat{Q}_\varepsilon(u)} \right\rangle.$$

Finally, we pass to the limit in \mathcal{D}' as $\varepsilon \rightarrow 0$. \square

3. THE GRADIENT FLOW AND POSITIVITY ([1])

We define the set of *positive maps* (in the class of equivariant Sobolev maps)

$$(23) \quad \mathcal{U}^{\text{Pos}} := \{u \in W_E^{1,2}(B_R; \mathbb{R}^n) \mid u(\overline{F_R}) \subset \overline{F}\}$$

and the set of *strongly positive maps*

$$(24) \quad \mathcal{U}_0^{\text{Pos}} := \{u \in W_E^{1,2}(B_R; \mathbb{R}^n) \mid u(F_R) \subset F\},$$

where $F_R = F \cap B_R$. Here $R > 0$ and clearly the sets \mathcal{U}^{Pos} and $\mathcal{U}_0^{\text{Pos}}$ depend on R .

We will utilize the gradient flow

$$(25) \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta u - W_u(u), & \text{in } B_R \times (0, \infty), \\ \frac{\partial u}{\partial \mathbf{n}} = 0, & \text{on } \partial B_R \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } B_R, \end{cases}$$

where $\partial/\partial \mathbf{n}$ is the normal derivative. We note that by Hypothesis 2

$$(26) \quad \langle -W_u(u), u \rangle \leq 0, \text{ for } |u| = M.$$

We will consider initial conditions in (25) satisfying in addition

$$(27) \quad \|u_0\|_{L^\infty(B_R; \mathbb{R}^n)} \leq M.$$

Since W is C^2 (cf. Hypothesis 1), the results in [12, Ch. 3, §3.3, §3.5] apply and provide a unique solution to (25) in $C(0, \infty; W_E^{1,2}(B_R; \mathbb{R}^n))$, which for $t > 0$, as a function of x , is in $C^{2+\alpha}(\overline{B_R}; \mathbb{R}^n)$, for some $0 < \alpha < 1$. Moreover, the solution satisfies the estimate

$$(28) \quad \|u(\cdot, t)\|_{L^\infty(B_R; \mathbb{R}^n)} \leq M, \text{ for } t \geq 0.$$

This follows from (26), (27), and by well-known invariance results [16, Ch. 14, §B].

Theorem 3.1 ([1]). *Let W be a C^2 potential satisfying Hypothesis 2. If $u_0 \in \mathcal{U}^{\text{Pos}}$ and $\|u_0\|_{L^\infty(B_R; \mathbb{R}^n)} \leq M$, then*

$$u(\cdot, t; u_0) \in \mathcal{U}^{\text{Pos}}, \text{ for } t \geq 0,$$

and, moreover,

$$u(\cdot, t; u_0) \in \mathcal{U}_0^{\text{Pos}}, \text{ for } t > 0, \text{ provided } u_0(\overline{F_R}) \cap F \neq \emptyset.$$

Proof. Let $u : B_R \rightarrow \mathbb{R}^n$ be an equivariant map. We will prove that u is a positive map if and only if

$$(29) \quad u(\overline{(\mathcal{P}_\gamma^+)_R}) \subset \overline{\mathcal{P}_\gamma^+}, \text{ for all } \gamma \in \Gamma,$$

where $(\mathcal{P}_\gamma^+)_R = \mathcal{P}_\gamma^+ \cap B_R$.

Suppose that (29) holds. Then

$$u(\overline{F_R}) = u(\cap_{\gamma \in \Gamma} \overline{(\mathcal{P}_\gamma^+)_R}) \subset \cap_{\gamma \in \Gamma} u(\overline{(\mathcal{P}_\gamma^+)_R}) \subset \cap_{\gamma \in \Gamma} \overline{\mathcal{P}_\gamma^+} = \overline{F}.$$

Hence, u is positive. Conversely, suppose that u is a positive equivariant map on B_R . Then, equivalently, u_e defined by

$$(30) \quad u_e(x) := \begin{cases} u(x), & \text{for } x \in B_R \\ 0, & \text{for } x \in \mathbb{R}^n \setminus B_R \end{cases}$$

is a positive equivariant map on \mathbb{R}^n . For any $g \in G$, we have from equivariance and positivity,

$$(31) \quad u_e(g(\overline{F})) = g(u_e(\overline{F})) \subset g(\overline{F}).$$

Now pick a $\gamma \in \Gamma$ and take an $x \in \mathcal{P}_\gamma^+$ and fix it. There is a $g \in G$, denoted by g_x , such that $x \in g_x(\overline{F})$ and $g_x(F)$ is also a fundamental region. Since for each fundamental region F' and for each reflection γ we have either $F' \subset \mathcal{P}_\gamma^+$ or $F' \subset -\mathcal{P}_\gamma^+$, we conclude that

$$(32) \quad g_x(\overline{F}) \subset \overline{\mathcal{P}_\gamma^+}.$$

Thus, by (31), $u_e(\overline{\mathcal{P}_\gamma^+}) \subset \overline{\mathcal{P}_\gamma^+}$, and so (29) follows.

Now consider (25) with $u_0 \in \mathcal{U}^{\text{Pos}}$. By the regularizing property of the equation the solution is classical for $t > 0$ and by (26) it exists globally in time and belongs to $C(0, +\infty; W_E^{1,2}(B_R; \mathbb{R}^n)) \cap C^1(0, +\infty; C^{2+\alpha}(\overline{B_R}; \mathbb{R}^n))$, for some $0 < \alpha < 1$ (see [12]). Consider a reflection $\gamma \in \Gamma$ and set

$$\begin{aligned} \zeta(x, t) &= \langle u(x, t, u_0), \eta_\gamma \rangle, \text{ on } B_R \times (0, \infty), \\ \zeta_0(x) &= \langle u_0(x), \eta_\gamma \rangle, \text{ on } B_R. \end{aligned}$$

By taking the inner product of equation (25) with η_γ , we obtain

$$(33) \quad \begin{cases} \frac{\partial \zeta}{\partial t} = \Delta \zeta + c\zeta, & \text{in } B_R \times (0, \infty), \\ \frac{\partial \zeta}{\partial \mathbf{n}} = 0, & \text{on } \partial B_R \times (0, \infty), \\ \zeta(\cdot, 0) = \zeta_0, \end{cases}$$

where we have set

$$c(x, t) = \frac{\langle W_u(u(x, t, u_0), \eta_\gamma) \rangle}{\zeta(x, t)}.$$

From the equivariance of $u(\cdot, t, u_0)$ and $W_u(\gamma u) = \gamma W_u(u)$ it follows that

$$(34) \quad \zeta(x, t) = -\zeta(\gamma x, t), \text{ in } B_R \times (0, \infty),$$

$$(35) \quad c(x, t) = c(\gamma x, t), \text{ in } B_R \times (0, \infty).$$

From the symmetry of W we also have that $u \in \pi_\gamma$ implies $W_u(u) \in \pi_\gamma$. From this we deduce

$$(36) \quad \langle W_u(u), \eta_\gamma \rangle = \langle u, \eta_\gamma \rangle \left\langle \int_0^1 W_{uu}(u + (s-1)\langle u, \eta_\gamma \rangle \eta_\gamma) \eta_\gamma ds, \eta_\gamma \right\rangle.$$

Thus, the coefficient $c(x, t)$ of ζ in (33) is bounded (actually continuous) on $B_R \times (0, \infty)$. Since u_0 is a positive map, we have $\zeta_0 \geq 0$ for $\langle x, \eta_\gamma \rangle \geq 0$. Therefore, for establishing positivity it is sufficient to show that $\zeta(x, t) \geq 0$, for $x \in B_R^+ = \{x \in B_R \mid \langle x, \eta_\gamma \rangle > 0\}$ and $t \geq 0$. We note that by (34) there holds $\zeta(x, t) = 0$ for $x \in \pi_\gamma \times [0, \infty)$, hence if ζ is a classical solution of (33), we have that $\zeta(x, t)$ is nonnegative on $B_R^+ \times [0, \infty)$ by the maximum principle. For general $\zeta_0 \in W^{1,2}(B_R)$ we approximate via mollification as in [7, §4.2, Thm. 2] and note that positivity and symmetry are preserved by the approximation process, rendering $\zeta_0^\varepsilon \in C^\infty(B_R) \cap L^\infty(B_R)$, with $\zeta_0^\varepsilon \rightarrow \zeta_0$ in $W^{1,2}(B_R)$. By the classical maximum principle, there holds that $\zeta^\varepsilon(x, t) \geq 0$ on $B_R^+ \times [0, \infty)$, and by continuous dependence for (33) in $W^{1,2}(B_R)$ [12, Thm. 3.4.1], we have that $\zeta^\varepsilon(\cdot, t) \rightarrow \zeta(\cdot, t)$ a.e. in B_R along subsequences $\varepsilon_n \rightarrow 0$, hence $\zeta(x, t) \geq 0$ a.e. Finally, since $\zeta(x, t) = 0$ for $x \in \pi_\gamma \times (0, \infty)$ and since $\zeta(\cdot, t) \in C^{2+\alpha}(\overline{B_R})$ for $t > 0$, the Hopf boundary lemma applies on the smooth part of ∂B_R^+ and renders

$$\zeta(x, t) > 0, \text{ in } B_R^+ \times (0, \infty),$$

unless $\zeta(x, t) \equiv 0$, hence unless $\zeta_0(x) \equiv 0$. But the hypothesis $u_0(\overline{F_R}) \cap F \neq \emptyset$ excludes this second option. \square

4. THE MINIMIZATION

Let $A^R := \{u \in W^{1,2}(B_R, \mathbb{R}^n) \mid u(\overline{F_R}) \subset \overline{F}\}$ and consider the minimization problem

$$\min_{A^R} J_{B_R}, \text{ where } J_{B_R}(u) = \int_{B_R} \left\{ \frac{1}{2} |\nabla u|^2 + W(u) \right\} dx.$$

We will argue first that the minimizer exists. We redefine $W(u)$ for $|u| \geq M+1$, so that the modified W is C^2 , satisfies $W(u) \geq c^2|u|^2$, for $|u| \geq M+1$ and a constant c , and also $W(gu) = W(u)$, for all $g \in G$. We still denote the modified potential by W and the modified functional by J_{B_R} . We note that the convexity of \overline{F} implies that A^R is convex and closed in $W_E^{1,2}(B_R; \mathbb{R}^n)$. The modified functional J_{B_R} satisfies all the properties required by the direct method and, as a result, a minimizer $v_R \in A^R$ exists.

Next we will show that as a consequence of Hypothesis 2 we can produce a minimizer $u_R \in A^R$, which in addition satisfies the estimate $|u_R(x)| \leq M$ (cf. (H3) in [3]). Due to this estimate, the values of W outside $\{|u| \leq M\}$ will not matter in the considerations in the rest of the paper and, therefore, the equation that will be solved is (1) with the original unmodified potential W . Set

$$(37) \quad u_R(x) = Pv_R(x),$$

where Pv equals the projection on the sphere $\{v \in \mathbb{R}^n \mid |v| = M\}$, for points outside the sphere ($Pv = Mv/|v|$), and equals the identity inside the sphere. Since P is a contraction with respect to the Euclidean norm in \mathbb{R}^n , it follows that $u_R \in W^{1,2}(B_R; \mathbb{R}^n)$, with $|\nabla u_R(x)| \leq |\nabla v_R(x)|$. Furthermore,

$$u_R(gx) = Pv_R(gx) = Pg v_R(x) = gPv_R(x) = gu_R(x),$$

hence $u_R \in W_E^{1,2}(B_R; \mathbb{R}^n)$. Clearly $u_R(\overline{F}) \subset \overline{F}$ and $|u_R(x)| \leq M$, for $x \in B_R$.

The fact that u_R is also a minimizer is a consequence of Hypothesis 2 and the following calculation.

$$\begin{aligned}
J_{B_R}(u) &\geq \int_{B_R} \left\{ \frac{1}{2} |\nabla v_R|^2 + W(v_R) \right\} dx, \text{ for } u \in A^R, \\
&\geq \int_{B_R} \left\{ \frac{1}{2} |\nabla u_R|^2 + W(v_R) \right\} dx \\
&= \int_{|v_R(x)| \leq M} \left\{ \frac{1}{2} |\nabla u_R|^2 + W(u_R) \right\} dx \\
&\quad + \int_{|v_R(x)| > M} \left\{ \frac{1}{2} |\nabla u_R|^2 + W(v_R) \right\} dx \\
&\geq \int_{|v_R(x)| \leq M} \left\{ \frac{1}{2} |\nabla u_R|^2 + W(u_R) \right\} dx \\
&\quad + \int_{|v_R(x)| > M} \left\{ \frac{1}{2} |\nabla u_R|^2 + W\left(M \frac{v_R}{|v_R|}\right) \right\} dx \\
&= \int_{B_R} \left\{ \frac{1}{2} |\nabla u_R|^2 + W(u_R) \right\} dx,
\end{aligned}$$

where the last inequality follows from Hypothesis 2.

We will be constructing the solution by taking the limit

$$(38) \quad u(x) = \lim_{R \rightarrow \infty} u_R(x).$$

For this purpose, we will need to show that the positivity constraint built in A^R does not affect the Euler–Lagrange equation, and we also need certain estimates, uniform in R , which in particular will imply that the solution is nontrivial.

Lemma 4.1. *Let u_R be as above. Then, for $R > 1$, the following hold.*

- (i) $J_{B_R}(u_R) \leq CR^{n-1}$, $\|u_R\|_{L^\infty(B_R; \mathbb{R}^n)} \leq M$, and $Q(u_R(x)) \leq \overline{Q}$, where $\overline{Q} := \max_{|u| \leq M} Q(u)$,
- (ii) $\Delta u_R - W_u(u_R) = 0$, in $W_{\text{loc}}^{1,2}(B_R; \mathbb{R}^n)$,
- (iii) u_R is positive (cf. (23)),
- (iv) $\Delta Q(u_R(x)) \geq 0$, in $W_{\text{loc}}^{1,2}(D_R)$, where $D_R := D \cap B_R$ (cf. Hypothesis 4).

Proof. For (i), define

$$u_{\text{aff}}(x) := \begin{cases} d(x; \partial D) a_1, & \text{for } x \in D_R \text{ and } d(x; \partial D) \leq 1, \\ a_1, & \text{for } x \in D_R \text{ and } d(x; \partial D) \geq 1, \end{cases}$$

and extend it equivariantly on B_R . Clearly, $u_{\text{aff}} \in A^R$. By the nonnegativity of W and a simple calculation,

$$(39) \quad 0 \leq J_{B_R}(u_R) \leq \min_{A^R} J_{B_R}(u) < J_{B_R}(u_{\text{aff}}) < CR^{n-1},$$

for some constant C independent of R . The rest of (i) is already known.

For (ii), by Theorem 3.1, we have $u(\cdot, t; u_R) \in A^R$, for $t \geq 0$. Since u_R is a global minimizer of J_{B_R} in A^R , and since $u(\cdot, t; u_R) \in C^1(0, \infty; C^{2+\alpha}(\overline{B_R}))$, a classical

solution to (25) for $t > 0$, we conclude from

$$(40) \quad \frac{d}{dt} J_{B_R}(u(\cdot, t)) = - \int_{B_R} |u_t|^2 dx$$

that $|u_t(x, t)| = 0$, for all $x \in B_R$ and $t > 0$. Hence, for $t > 0$, $u(\cdot, t)$ is satisfying

$$(41) \quad \Delta u(x, t) - W_u(u(x, t)) = 0.$$

By taking $t \rightarrow 0+$ and utilizing the continuity of the flow in $W^{1,2}(B_R; \mathbb{R}^n)$ at $t = 0$, $u(\cdot, \cdot; u_R) \in C([0, \infty); W^{1,2}(B_R; \mathbb{R}^n))$, we obtain (ii).

Since (iii) is already known, we go on to (iv) where we obtain from (41), for $t > 0$,

$$\begin{aligned} 0 &= \langle Q_u(u(x, t)), \Delta u(x, t) \rangle - \langle Q_u(u(x, t)), W_u(u(x, t)) \rangle \\ &= \langle \hat{Q}_u(u(x, t) - a_1), \Delta(u(x, t) - a_1) \rangle - \langle Q_u(u(x, t)), W_u(u(x, t)) \rangle \end{aligned}$$

where $Q(u) = \hat{Q}(u - a_1)$, while using (12) we continue to obtain

$$\begin{aligned} 0 &\leq \Delta \hat{Q}(u(x, t) - a_1) - \langle Q_u(u(x, t)), W_u(u(x, t)) \rangle \\ &= \Delta Q(u(x, t)) - \langle Q_u(u(x, t)), W_u(u(x, t)) \rangle \\ (42) \quad &\leq \Delta Q(u(x, t)), \end{aligned}$$

by Theorem 3.1, utilizing $u_R \in \mathcal{U}^{\text{Pos}}$, from which it follows that $u(\overline{D_R}, t) \subset D$, and by Hypothesis 4, particularly (10).

Thus, by the second remark following Lemma 2.1, we have

$$(43) \quad \Delta Q(u(x, t)) \geq 0, \text{ in } W_{\text{loc}}^{1,2}(D_R), \text{ for } t > 0,$$

or, equivalently,

$$(44) \quad \int_{D_R} \nabla Q(u(x, t)) \nabla \phi(x) dx \leq 0, \text{ for all } \phi \geq 0, \phi \in W_{\text{loc}}^{1,2}(D_R).$$

We will argue that

$$(45) \quad \nabla Q(u(\cdot, t)) \rightarrow \nabla Q(u_R(\cdot)), \text{ weakly in } L^2(B_R), \text{ as } t \rightarrow 0,$$

via which the proof of (iv) will be concluded. We know that

$$(46) \quad \begin{cases} u(\cdot, t; u_R) \rightarrow u_R, \text{ in } W^{1,2}(B_R; \mathbb{R}^n), \text{ as } t \rightarrow 0, \\ \|u(\cdot, t; u_R)\|_{L^\infty(B_R; \mathbb{R}^n)} \leq M. \end{cases}$$

Hence,

$$Q(u(\cdot, t; u_R)) \rightarrow Q(u_R), \text{ in } L^2(B_R), \text{ as } t \rightarrow 0,$$

since Q_u can be taken globally bounded. Thus,

$$\nabla Q(u(\cdot, t; u_R)) \rightarrow \nabla Q(u_R), \text{ in } \mathcal{D}'(B_R), \text{ as } t \rightarrow 0.$$

However, $\|\nabla Q(u(\cdot, t; u_R))\|_{L^2(B_R)} < C$ by (46). Therefore (45) is established and the proof is complete. \square

The consideration in Lemma 4.1, particularly (40), together with the fact that u_R is a global minimizer, show that $u(\cdot, t; u_R)$ is an equilibrium of (25) for $t > 0$, that is, a time-independent solution satisfying in addition the boundary condition $\partial u / \partial \mathbf{n} = 0$. We can therefore replace u_R with this equilibrium which satisfies all the properties of Lemma 4.1 and also is in $C^{2+\alpha}(\overline{B_R}; \mathbb{R}^n)$.

Corollary 4.2. *We may assume that $u_R \in C^{2+\alpha}(\overline{B_R}; \mathbb{R}^n)$ is an equilibrium of (25) that satisfies all the properties of Lemma 4.1. Then,*

$$u_R(\overline{F_R}) \cap \overline{F} \neq \emptyset \quad \text{implies} \quad u_R \in \mathcal{U}_0^{\text{Pos}}.$$

Proof. This follows from Theorem 3.1 and the fact that u_R is a time-independent solution of (25). \square

5. THE COMPARISON FUNCTION σ ([1])

We prove three lemmas leading to the construction of a map σ that will play a major role in the derivation of the uniform estimates in R in the following section. We let χ_A be the characteristic function of a set A .

Given numbers $l, \lambda > 0$, set $L = l + \lambda$ and let $\varphi = \chi_{\overline{B_l}} \varphi_1 + \chi_{\overline{B_L} \setminus \overline{B_l}} \varphi_2$, where $\varphi_1 : \overline{B_l} \rightarrow \mathbb{R}$, $\varphi_2 : \overline{B_L} \setminus B_l \rightarrow \mathbb{R}$ are defined by

$$(47) \quad \begin{cases} \Delta \varphi_1 = c^2 \varphi_1, & \text{in } B_l, \\ \varphi_1 = \bar{q}, & \text{on } \partial B_l, \end{cases}$$

and

$$(48) \quad \begin{cases} \Delta \varphi_2 = 0, & \text{in } B_L \setminus \overline{B_l}, \\ \varphi_2 = \bar{q}, & \text{on } \partial B_l, \\ \varphi_2 = \overline{Q}, & \text{on } \partial B_L, \end{cases}$$

where c , \bar{q} , and M below are the constants defined in Hypotheses 1 and 2 and

$$(49) \quad \overline{Q} = \max_{|u| \leq M} Q(u),$$

(see Hypothesis 2 and (i) of Lemma 4.1). The map φ is radial, that is, $\varphi_j(x) = \phi_j(|x|)$, for $j = 1, 2$. Classical properties of Bessel functions imply that $\phi_1 : [0, l] \rightarrow \mathbb{R}$ is positive and increasing together with the first derivative ϕ_1' . The function $\phi_2 : [l, L] \rightarrow \mathbb{R}$ is increasing with decreasing first derivative ϕ_2' , by explicit calculation.

Lemma 5.1. *The following hold.*

(i) *The function $\phi_1'(l)$ is strictly increasing for $l \in (0, +\infty)$ and*

$$(50) \quad \lim_{l \rightarrow +\infty} \phi_1'(l) = c\bar{q}.$$

(ii) *There exists a strictly increasing function $h : (0, +\infty) \rightarrow (0, +\infty)$ such that*

$$(51) \quad \phi_1(r) \leq e^{h(l)(r-l)} \phi_1(l), \quad \text{for } r \in [0, l],$$

and $\lim_{l \rightarrow +\infty} h(l) = c$.

(iii) *There is a constant C_0 , independent of l , such that*

$$(52) \quad \phi_1''(r) \leq C_0, \quad \text{for } r \in [0, l].$$

Proof. Statements (i) and (ii) are proved in [8, Lemma 2.4]. For (iii) note that

$$(53) \quad \phi_1'' = c^2 \phi_1 - \frac{n-1}{r} \phi_1' \leq c^2 \psi_1 \leq c^2 \bar{q},$$

since ϕ_1 is increasing and bounded by \bar{q} . \square

An explicit computation yields, for $r \in [l, L]$,

$$(54) \quad \phi'_2(r) = \begin{cases} \frac{\bar{Q} - \bar{q}}{r \log(L/l)}, & \text{for } n = 2, \\ (n-2) \frac{l^{n-2}(\bar{Q} - \bar{q})}{r^{n-1}(1 - (l/L)^{n-2})}, & \text{for } n > 2. \end{cases}$$

Lemma 5.2. *The following hold.*

(i) *Let the ratio l/L be fixed. Then,*

$$(55) \quad \lim_{l \rightarrow +\infty} \phi'_2(l) = 0.$$

(ii) *Let the difference $L - l = \lambda$ be fixed. Then, $\phi'_2(l)$ is a decreasing function of $l \in (0, +\infty)$ and*

$$(56) \quad \lim_{l \rightarrow +\infty} \phi'_2(r) = \frac{\bar{Q} - \bar{q}}{\lambda}, \text{ for } r \in [l, l + \lambda].$$

Moreover, there exists a constant C_0 , independent of $l \in [1, +\infty)$, such that

$$(57) \quad |\phi''_2(r)| \leq \frac{C_0}{l}, \text{ for } r \in [l, l + \lambda].$$

Proof. Statement (i) is a straightforward consequence of (54). We prove (ii) for $n > 2$. The case $n = 2$ is similar. To show that $\phi'_2(l)$ is decreasing, we prove that the map $f(l) = l(1 - (l/(l + \lambda))^{n-2})$ is increasing. Setting $\xi = l/(l + \lambda)$ we have

$$f'(l) = d(\xi) := 1 - (n-1)\xi^{n-2} + (n-2)\xi^{n-1}, \text{ for } \xi \in [0, 1],$$

and $f'(l) > 0$, for $l \in (0, +\infty)$, follows from $d(0) = 1$, $d(1) = 0$, and $d'(\xi) < 0$, for $\xi \in (0, 1)$. The limit (56) follows from (54). The last statement of the lemma follows from

$$\phi''_2(r) = -(n-1) \frac{l^{n-1}}{r^n} \phi'_2(l). \quad \square$$

Let φ be as before and let $\delta > 0$ be a small number. Denote by $\vartheta : B_{l+\delta} \setminus \overline{B_{l-\delta}} \rightarrow \mathbb{R}$ the solution of the problem

$$(58) \quad \begin{cases} \Delta \vartheta = 0, & \text{in } B_{l+\delta} \setminus \overline{B_{l-\delta}}, \\ \vartheta = \varphi, & \text{on } \partial(B_{l+\delta} \setminus \overline{B_{l-\delta}}). \end{cases}$$

We have $\vartheta(x) = \theta(|x|)$, where $\theta : [l - \delta, l + \delta] \rightarrow \mathbb{R}$ satisfies

$$(59) \quad \theta'(r) = \begin{cases} \frac{\phi_2(l + \delta) - \phi_1(l - \delta)}{r \log \frac{l+\delta}{l-\delta}}, & \text{for } n = 2, \\ (n-2) \frac{(l - \delta)^{n-2}(\phi_2(l + \delta) - \phi_1(l - \delta))}{r^{n-1}(1 - (\frac{l-\delta}{l+\delta})^{n-2})}, & \text{for } n > 2. \end{cases}$$

Lemma 5.3. *There exist positive constants l_0 , λ , δ , $\bar{q}' < \bar{q}$, δ' , μ , such that $l \geq l_0$, $L = l + \lambda$ implies*

- (i) $\phi'_1(l) > \phi'_2(l) + \mu$,
- (ii) $\vartheta < \varphi$, in $B_{l+\delta} \setminus \overline{B_{l-\delta}}$,

(iii) *The map $\sigma : \overline{B_L} \rightarrow \mathbb{R}$ defined by $\sigma = \chi_{B_{l-\delta} \cup (\overline{B_L} \setminus \overline{B_{l+\delta}})}\varphi + \chi_{\overline{B_{l+\delta}} \setminus B_{l-\delta}}\vartheta$ satisfies*

$$(60) \quad \sigma \leq \bar{q}' < \bar{q}, \text{ in } \overline{B_{l+\delta'}}.$$

Proof. Letting the ratio $\rho = l/L$ be fixed, then (50) and (55) imply that there is an l_0 such that (i) holds for $l = l_0$ and some $\mu > 0$. Fixing $\lambda = l_0((l/\rho) - 1)$, then (i) holds for all $l \geq l_0$. This follows from Lemmas 5.1 and (ii) of Lemma 5.2, which imply that $\phi_1'(l)$ is increasing and $\phi_2'(l)$ is decreasing for fixed λ . From (59), the relation

$$\phi_2(l + \delta) - \phi_1(l - \delta) = (\phi_2'(l) + \phi_1'(l))\delta + o(\delta),$$

which holds uniformly in l since $\phi_1(l) = \phi_2(l) = \bar{q}$, and

$$\log \frac{l + \delta}{l - \delta} = 2\frac{\delta}{l} + o(\delta), \quad \left(\frac{l - \delta}{l + \delta}\right)^{n-2} = 1 - 2(n-2)\frac{\delta}{l} + o(\delta),$$

it follows that

$$(61) \quad \left| \theta'(r) - \frac{1}{2}(\phi_2'(l) + \phi_1'(l)) \right| \leq C\delta, \text{ for } r \in [l - \delta, l + \delta],$$

$$(62) \quad |\theta''| \leq \frac{C}{l}, \text{ for } r \in [l - \delta, l + \delta]$$

for some constant $C > 0$, independent of $l \in [l_0, +\infty)$. From (i) and (61), and the bounds on ϕ_1'' , ϕ_2'' , θ'' , it follows that there is a small $\delta > 0$, independent of $l \in [l_0, +\infty)$, such that

$$\begin{cases} \theta'(r) < \phi_1'(r), & \text{for } r \in [l - \delta, l], \\ \theta'(r) > \phi_2'(r), & \text{for } r \in [l, l + \delta]. \end{cases}$$

This and $\theta(l - \delta) = \phi_1(l - \delta)$, $\theta(l + \delta) = \phi_2(l + \delta)$, prove (ii). The existence of the number $\bar{q}' < \bar{q}$ and $0 < \delta' < \delta$, independent of $l \in [l_0, +\infty)$, follows by the same arguments and from the existence of the limits (50) and (56). \square

6. UNIFORM ESTIMATES IN R

In this section we will make use of special notation. We denote by $B_R(x_R)$ the ball of radius $R > 0$ centered at x_R . As before, B_R denotes the ball of radius $R > 0$ centered at the origin and $D_{4R} = D \cap B_{4R}$, with x_R a point in D_{4R} such that $B_R(x_R) \subset D_{4R}$. The function u_{4R} is the minimizer for the functional $J_{B_{4R}}$ in Corollary 4.2.

Set

$$(63) \quad v_R(x) := \frac{Q(u_{4R}(x)) - \bar{q}/2}{\bar{Q} - \bar{q}/2}, \text{ for } x \in B_R(x_R),$$

where \bar{q} as in Hypothesis 1, \bar{Q} as in Lemma 4.1, Hypothesis 2, with $\bar{Q} > \bar{q}/2$. We will also rescale the dependent variable via $y = (x - x_R)/R$ and define

$$(64) \quad \hat{v}_R(y) := v_R(Ry + x_R) = v_R(x), \text{ for } y \in \hat{B}_1,$$

where $\hat{B}_1 := \{y \in \mathbb{R}^n \mid |y| < 1\}$, $B_1 := \{x \in \mathbb{R}^n \mid |x| < 1\}$, and $B_R^+(x_R) := \{x \in B_R(x_R) \mid v_R(x) \geq 0\}$, $B_R^-(x_R) := \{x \in B_R(x_R) \mid v_R(x) \leq 0\}$, and analogously, $\hat{B}_1^+ := \{y \in \hat{B}_1 \mid \hat{v}_R(y) \geq 0\}$, $\hat{B}_1^- := \{y \in \hat{B}_1 \mid \hat{v}_R(y) \leq 0\}$. Notice that \hat{B}_1^+ , \hat{B}_1^- depend on R .

By definition

$$(65) \quad Q(u_{4R}(x)) \geq \frac{\bar{q}}{2}, \text{ on } B_R^+(x_R).$$

By positivity ((iii) of Lemma 4.1) and equivariance, there holds $u_{4R}(B_R(x_R)) \subset u_{4R}(D_{4R}) \subset \overline{D}$. Hence,

$$(66) \quad W(u_{4R}(x)) \geq \varepsilon_0(\bar{q}) > 0, \text{ on } B_R^+(x_R),$$

since a_1 is the unique zero of W in \overline{D} (Hypotheses 3, 4).

Lemma 6.1. *The following estimate holds for the Lebesgue measure of \hat{B}_1^- .*

$$(67) \quad |\hat{B}_1^-| \geq |\hat{B}_1| - \frac{C}{\varepsilon_0(\bar{q})R},$$

where C is a constant depending only on the constant C in (i) of Lemma 4.1 and the dimension n .

Proof. We have

$$\begin{aligned} CR^{n-1} &\geq \int_{B_{4R}} W(u_{4R}(x)) \, dx \quad (\text{by (i) of Lemma 4.1}) \\ &\geq \int_{B_R^+(x_R)} W(u_{4R}(x)) \, dx \quad (\text{by } W \geq 0) \\ &\geq \varepsilon_0(\bar{q}) |B_R^+(x_R)| \quad (\text{by (66)}). \end{aligned}$$

Therefore,

$$\frac{C}{R} \geq \varepsilon_0(\bar{q}) \frac{|B_R^+(x_R)|}{R^n} = \varepsilon_0(\bar{q}) |\hat{B}_1^+|,$$

hence

$$|\hat{B}_1^-| = |\hat{B}_1| - |\hat{B}_1^+| \geq |\hat{B}_1| - \frac{C}{\varepsilon_0(\bar{q})R}. \quad \square$$

Remark. The lemma above is a direct consequence of the basic integral estimate in (i) Lemma 4.1 and the assumption that in D the potential has a unique zero. Estimate (67) states that the minimizer $u_{4R}(x)$ on a set of large measure in $B_R(x_R)$ is close to a_1 , the zero of W , for $R \rightarrow \infty$.

The point in the next lemma is that the subharmonicity of $Q(u_{4R}(x))$ in D (by (iv) of Lemma 4.1) via a classical result of De Giorgi (see Appendix) allows us to obtain a pointwise estimate in the ball $B_{R/2}(x_R)$ of half the radius.

Lemma 6.2. *Fix a $\delta \in (0, 1)$. Then, for R large enough such that*

$$(68) \quad 1 - \frac{1}{\varepsilon_0(\bar{q})} \frac{C}{R} \frac{1}{c_0} \geq 1 - \delta,$$

we have the estimate

$$(69) \quad \sup_{B_{R/2}(x_R)} Q(u_{4R}(x)) \leq \frac{\bar{q}}{2} + \mu(1 - \delta) \left(\overline{Q} - \frac{\bar{q}}{2} \right),$$

where $\mu(\cdot)$ is defined in the Appendix, with $\mu(1 - \delta) < 1$.

Here C is the constant in Lemma 6.1 and c_0 is the volume of the unit ball in \mathbb{R}^n .

Proof. Note that $\Delta_y \hat{v}_R \geq 0$ in \hat{B}_1 and $\hat{v}_R \leq 1$, in \hat{B}_1 , by (iv) and (i) of Lemma 4.1 respectively, and moreover

$$\frac{|\hat{B}_1^-|}{|\hat{B}_1|} \geq 1 - \frac{1}{\varepsilon_0(\bar{q})} \frac{C}{R} \frac{1}{c_0} \geq 1 - \delta$$

by (67). Hence, by the lemma in the Appendix,

$$\sup_{\hat{B}_{1/2}} \hat{v}_R(y) \leq \mu(1 - \delta) < 1,$$

which is equivalent to (69). \square

Next we will iterate. The number δ is fixed in Lemma 6.2 and we select k as the minimal integer with the property

$$(70) \quad \frac{\bar{q}}{2} + (\mu(1 - \delta))^k \left(\bar{Q} - \frac{\bar{q}}{2} \right) < \bar{q}.$$

Clearly k depends only on δ . Finally we choose $R_0 = R_0(\delta)$ such that

$$(71) \quad 1 - \frac{1}{\varepsilon_0(\bar{q})} \frac{C}{R} \frac{1}{|\hat{B}_{1/2^k}|} \geq 1 - \delta, \text{ for } R \geq R_0,$$

with C as in Lemma 6.2.

From now on, R , in the definition of A^R and in the definition of the minimizer u_R , is assumed to satisfy (71), and free otherwise. For such an R we define

$$\begin{cases} \bar{Q}_0 := \bar{Q}, \\ \bar{Q}_i := \frac{\bar{q}}{2} + \mu(a^*) \left(\bar{Q}_{i-1} - \frac{\bar{q}}{2} \right), \\ v_i(x) := \frac{Q(u_{4R}(x)) - \bar{q}/2}{\bar{Q}_{i-1} - \bar{q}/2}, \text{ for } x \in B_{R/2^i}(x_R), \\ \hat{v}_i(y) := v_i(Ry), \text{ for } y \in \hat{B}_{1/2^i}, \end{cases}$$

for $i = 1, 2, \dots, k$ and $a^* = 1 - \delta$.

We notice that (71) implies all the corresponding inequalities for $i = 1, 2, \dots, k$ and, particular, (68).

Lemma 6.3. *For an integer $k = k(\delta)$, as in (70), and for $R \geq R_0(\delta)$, as in (71), the following estimate holds.*

$$(72) \quad \sup_{B_{R/2^k}(x_R)} Q(u_{4R}(x)) \leq \frac{\bar{q}}{2} + (\mu(a^*))^k \left(\bar{Q} - \frac{\bar{q}}{2} \right) < \bar{q}.$$

Proof. We make the simple observation that

$$(73) \quad \sup_{B_{R/2^i}(x_R)} Q(u_{4R}(x)) \leq \frac{\bar{q}}{2} + (\mu(a^*))^i \left(\bar{Q} - \frac{\bar{q}}{2} \right)$$

holds for $i = 1, 2, \dots, k$.

We note that for $i = 1$ this is just (69). Let us establish (73) for $i = 2$. We may assume that $k \geq 3$ since otherwise we have the estimate we need, hence

$\overline{Q}_1 > \bar{q}/2$. We have $Q(u_{4R}(x)) \leq \overline{Q}_1$, on $B_{R/2}(x_R)$, by (69), and $Q(u_{4R}(x)) \geq \bar{q}/2$, on $B_{R/2}^+(x_R)$, by definition. Hence,

$$\begin{aligned} CR^{n-1} &\geq \int_{B_R^+(x_R)} W(u_{4R}(x)) \, dx \\ &\geq \int_{B_{R/2}^+(x_R)} W(u_{4R}(x)) \, dx \quad (\text{cf. Proof of Lemma 6.1}) \\ &\geq \varepsilon_0(\bar{q}) |B_{R/2}^+(x_R)| \quad (\text{by (66)}). \end{aligned}$$

Therefore,

$$\frac{C}{R} \geq \varepsilon_0(\bar{q}) \frac{|B_{R/2}^+(x_R)|}{R^n} = \varepsilon_0(\bar{q}) |\hat{B}_{1/2}^+|,$$

hence,

$$|\hat{B}_{1/2}^-| = |\hat{B}_{1/2}| - |\hat{B}_{1/2}^+| \geq |\hat{B}_{1/2}| - \frac{C}{\varepsilon_0(\bar{q})R}.$$

It follows that

$$(74) \quad \frac{|\hat{B}_{1/2}^-|}{|\hat{B}_{1/2}|} \geq 1 - \frac{C}{\varepsilon_0(\bar{q})R|\hat{B}_{1/2}|} \geq 1 - \delta,$$

by (71).

On the other hand, $\Delta_y \hat{v}_2(y) \geq 0$ in $\hat{B}_{1/2}$ and $\hat{v}_2 \leq 1$, in $\hat{B}_{1/2}$, hence, by the lemma in the Appendix,

$$\sup_{\hat{B}_{1/2}^2} \hat{v}_2(y) \leq \mu(a^*),$$

which equivalently gives

$$\sup_{B_{R/2^2}(x_R)} Q(u_{4R}(x)) \leq \frac{\bar{q}}{2} + \mu(a^*) \left(\overline{Q}_1 - \frac{\bar{q}}{2} \right),$$

or

$$\sup_{B_{R/2^2}(x_R)} Q(u_{4R}(x)) \leq \frac{\bar{q}}{2} + \mu(a^*)^2 \left(\overline{Q} - \frac{\bar{q}}{2} \right).$$

By repeating this process for $i = 3, \dots, k$, we obtain (72). \square

So far we have established that

$$(75) \quad \sup_{B_{R^*}(x_R)} Q(u_{4R}(x)) \leq \bar{q},$$

where $R^* = R/2^k$, for $R \geq R_0$, and an integer k independent of R . Utilizing the comparison function σ in Section 5 it is possible to show that the ball $B_{R^*}(x_R)$ in the supremum in (75) can be replaced by a large set D_R^* which includes all of D_{4R} with the exception of a strip along the boundary ∂D of width d_0 independent of R , for $R \geq R_0$, that is,

$$(76) \quad D_R^* \supset \{x \in D_{4R} \mid d(x, \partial D_{4R}) \geq d_0\},$$

for some $d_0 > 0$, which depends on l_0 in Lemma 5.3.

Lemma 6.4. *The following estimate holds.*

$$(77) \quad \sup_{D_R^*} Q(u_{4R}) \leq \bar{q},$$

where D_R^* has the properties stated above.

Proof. First we note that by Hypotheses 1, 4,

$$\langle Q_u(u), W_u(u) \rangle \geq c^2 Q(u), \text{ for } |u - a_1| \leq \bar{q},$$

which implies, via Lemma 2.1, (ii) of Lemma 4.1, and (75), the estimate

$$(78) \quad \Delta Q(u_{4R}) \geq \langle \Delta u_{4R}, Q_u(u_{4R}) \rangle = \langle W_u(u_{4R}), Q_u(u_{4R}) \rangle \geq c^2 Q(u_{4R}),$$

in $W_{\text{loc}}^{1,2}(B_{R^*}(x_R))$.

Next we refer to Section 5. Consider a ball $B_l(\xi)$, tangent to $\partial B_{R^*}(x_R)$ and with its center ξ inside $B_{R^*}(x_R)$, and also consider the concentric ball $B_L(\xi)$. Notice that $B_l(\xi)$ is the translation of B_l and $B_L(\xi)$ the translation B_L . Similarly consider the translations of $\varphi_1, \varphi_2, \vartheta$, which we still denote by the same symbols.

We now observe by (47), (75), and (78), that

$$\begin{cases} \Delta \varphi_1 = c^2 \varphi_1, & \text{in } B_l(\xi), \\ \varphi_1 = \bar{q}, & \text{on } \partial B_l(\xi), \end{cases} \quad \begin{cases} \Delta Q(u_{4R}) \geq c^2 Q(u_{4R}), & \text{in } B_l(\xi), \\ Q(u_{4R}) \leq \bar{q}, & \text{on } \partial B_l(\xi), \end{cases}$$

hence, by the maximum principle for $W^{1,2}$ solutions (see [9]), we have

$$(79) \quad Q(u_{4R}) \leq \varphi_1, \text{ in } B_l(\xi).$$

Also, by (48), (i) and (iv) of Lemma 4.1, and (75),

$$\begin{cases} \Delta \varphi_2 = 0, & \text{in } B_L(\xi) \setminus \overline{B_l(\xi)}, \\ \varphi_2 = \bar{q}, & \text{on } \partial B_l(\xi), \\ \varphi_2 = \overline{Q}, & \text{on } \partial B_L(\xi), \end{cases} \quad \begin{cases} \Delta Q(u_{4R}) \geq 0, & \text{in } B_L(\xi), \\ Q(u_{4R}) \leq \varphi_2, & \text{on } \partial(B_L(\xi) \setminus \overline{B_l(\xi)}), \end{cases}$$

hence,

$$(80) \quad Q(u_{4R}) \leq \varphi_2, \text{ in } B_L(\xi) \setminus \overline{B_l(\xi)}.$$

We deduce therefore by Lemma 5.3 that

$$(81) \quad \begin{cases} Q(u_{4R}) \leq \varphi, & \text{in } B_L(\xi), \\ Q(u_{4R}) \leq \vartheta, & \text{in } B_{l+\delta}(\xi) \setminus \overline{B_{l-\delta}(\xi)}, \\ Q(u_{4R}) \leq \sigma \leq \bar{q}' < \bar{q}, & \text{in } \overline{B_{l+\delta'}(\xi)} \end{cases}$$

Thus, we see from (iii) of (81) that the estimate (75) holds on a set larger than $B_{R^*}(x_R)$. Clearly, by repeating this process we obtain (77). \square

We are now able to finish the proof of the theorem.

Proof of Theorem 1.1. First note that if $q : D_R^* \rightarrow \mathbb{R}$ is the solution to

$$(82) \quad \Delta q = c^2 q, \text{ in } D_R^*, q = \bar{q}', \text{ on } D_R^*,$$

then,

$$(83) \quad q(x) \leq K e^{-kd(x, \partial D_R^*)}, \text{ for } x \in D_R^*,$$

for positive constants K, k , independent of R .

Indeed, by the maximum principle, there holds $q \leq \bar{q}'$. It follows that if φ is the solution of equation (82) on the ball with center x and radius $d(x, \partial D_R^*)$ and with boundary condition $\varphi = \bar{q}'$, we have $q \leq \varphi$. This and the estimate (51) in (ii) of Lemma 5.1 imply (83). Moreover, we note that if $d_0 > 0$ is as in (76), then

$$(84) \quad q(x) \leq K e^{-kd(x, \partial D_{4R})}, \text{ in } B_{d_0}(x) \subset D_{4R},$$

since $d(x, \partial D_{4R}) \leq d(x, \partial D_R^*) + d_0$. This last inequality follows from (76) with a new constant K .

Now utilizing (75) and $\Delta Q(u_{4R}) \geq c^2 Q(u_{4R})$, in $W_{\text{loc}}^{1,2}(D_R^*)$, by (78), we obtain by comparing with (82) that $Q(u_{4R}(x)) \leq q(x)$, for $x \in D_R^*$, and, therefore, by (84) and (9c),

$$(85) \quad |u_{4R}(x) - a_1| \leq K e^{-kd(x, \partial D_{4R})}.$$

The uniform bound in (i) of Lemma 4.1 and elliptic regularity, via a diagonal argument, allow us to pass to the limit along a subsequence in R and capture a function

$$u(x) = \lim_{R_n \rightarrow \infty} u_{R_n}(x).$$

The uniform bound (85) implies that the limit function satisfies the exponential estimate in the theorem and is also a solution to

$$\Delta u - W_u(u) = 0, \text{ in } \mathbb{R}^n,$$

by (ii) of Lemma 4.1 (and the comment after its proof). Clearly, also $u \in \mathcal{U}^{\text{Pos}}$.

Finally we argue the strong positivity for $u(x)$ with respect to D . Take now an open connected set $U \subset D$ which contains some points far enough from ∂D so that by the exponential estimate there holds $u(U) \cap D \neq \emptyset$. Define $\Gamma' := \{\gamma \in \Gamma \mid \pi_\gamma \cap D \neq \emptyset\}$; then, $D = \cap_{\gamma \in \Gamma \setminus \Gamma'} \mathcal{P}_\gamma^+$ (see Lemmas 2.1 and 5.1 in [1]). As in the second part of the proof of Theorem 3.1, particularly (33), set $z(x) = \langle u(x), \eta_\gamma \rangle$, for $x \in U$ and $\gamma \in \Gamma \setminus \Gamma'$. Then,

$$\begin{cases} \Delta z + cz = 0, & \text{in } U, \\ z \geq 0, & \text{in } U. \end{cases} \quad (\text{by positivity})$$

By a well-known variant of the strong maximum principle, there holds $z > 0$ in U , unless $z \equiv 0$. Triviality is excluded by $u(U) \cap D \neq \emptyset$ above. From this, strong positivity follows.

The proof is complete. \square

APPENDIX

We state a special case of the *De Giorgi oscillation lemma* which was originally established for general elliptic operators $\mathcal{L}u := \text{div}(A(x)\nabla u)$, with bounded, measurable coefficients (see [4, p. 195]).

Lemma (De Giorgi). *Consider the ball $\hat{B}_1 = \{y \in \mathbb{R}^n \mid |y| \leq 1\}$ and a function $\hat{v} = \hat{v}(y)$ which satisfies*

- (i) $\Delta_y \hat{v} \geq 0$, in $W_{\text{loc}}^{1,2}(\hat{B}_1)$,
- (ii) $\hat{v} \leq 1$, in \hat{B}_1 ,
- (iii) $|\hat{B}_1^-|/|\hat{B}_1| \geq a^* > 0$, for $\hat{B}_1^- = \{y \in \hat{B}_1 \mid \hat{v}(y) \leq 0\}$,

where $|\Omega|$ is the Lebesgue measure of the set Ω . Then,

$$\sup_{\hat{B}_{1/2}} \hat{v} \leq \mu(a^*) < 1.$$

Notice that the statement of the lemma is invariant under the scaling $y \rightarrow \lambda y$, for $\lambda > 0$, hence \hat{B}_1 can be replaced by \hat{B}_λ and $\hat{B}_{1/2}$ by $\hat{B}_{\lambda/2}$, without affecting $\mu(a^*)$.

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